A MATHEMATICAL MODEL OF THE DYNAMICS OF BLOOD FLOW IN THE LARGE ARTERIES

MADUEME, PRAISE-GOD U., MBAH GODWIN C.E.
Department of Mathematics, University of Nigeria, Nsukka
praise-god.madueme@unn.edu.ng, godwin.mbah@unn.edu.ng,
+2348065577024, +2348034198454

ABSTRACT

This research studied the flow of blood through the arteries with emphasis on the large arteries considering it with elastic wall. In this work, blood was considered as a Newtonian fluid and the artery being a cylindrical tube. Given that the artery is cylindrical in shape with the motion of the blood being pulsatile, the model was generated using the Navier-Stokes' equation. In the analysis of the study, it was discovered that the graphical representations for the radial and axial velocities in blood flow is sinusoidal which is valid since the elastic arterial walls respond to the pulsatile nature of the heart. The study also shows how blood pressure increases from the aorta, through the arteries (systemic circulation), and begins to drop as the arteries divides into smaller arterioles up to the capillaries and veins (pulmonary circulation). Its understanding goes a long way to affect health care spending with its effect on the GDP and inflation of a nation's economy.

Keywords: Pulsatile, Newtonian, Navier-Stokes, laminar

1. INTRODUCTION

The study of the behavior of blood flow in the blood vessels provides understanding of blood flow dynamics. The aorta and arteries have a low resistance to blood flow compared with the arterioles and capillaries. When the ventricle contracts, a volume of blood is rapidly ejected into the arterial vessels. Since the outflow to the arteriole is relatively slow because of their high resistance to flow, the arteries are inflated to accommodate the extra blood volume. During diastole, the elastic recoil of the arteries forces the blood out into the arterioles. Thus, the elastic properties of the arteries help to convert the pulsatile flow of blood from the heart into a more continuous flow through the rest of the circulation. Hemodynamics is a term used to describe the mechanisms that affect the dynamics of blood circulation. The cyclic nature of the heart pump creates pulsatile conditions in all arteries. The heart ejects and fills with blood in alternating cycles called *systole* and *diastole*. Blood is pumped out of the heart during systole. The heart rests during diastole, and no blood is ejected. The flow out of the heart is intermittent, going to zero when the aortic valve is closed. The aorta, the large artery taking blood out of the heart, serves as a compliance chamber that provides a reservoir of high pressure during diastole as well as systole. Thus the blood pressure in most arteries is pulsatile, yet does not go to zero during diastole. In contrast, the flow is zero or even reversed during diastole in some arteries such as the external carotid, brachial, and femoral arteries. These arteries have a high downstream resistance during rest and the flow is essentially on/off with each cycle. In other arteries such as the internal carotid or the renal arteries, the flow can be high during diastole if the downstream resistance is low. The flow in these arteries is more uniform (Martini, 1995; Thibodeau and Patton, 1999).

Most researches have studied the blood flow in the arteries and veins. One of the motivations to study the blood flow was to understand the conditions that may contribute to blood related diseases such as mumurs, high blood pressure etc. Past studies indicated that one of the reasons for a person having hypertension is when the blood vessel becomes narrow. Arteries contain more muscles than comparably sized veins. Large arteries stretch when the pressure of the blood rises during systole and recoil during diastole. The elastic recoil of the walls helps to produce a smoother flow of blood in the smaller arteries and arterioles. However, the result is a cardiac cycle-dependent artery diameter. Smaller arteries and arterioles are less elastic than larger arteries and contain a proportionally thicker layer of smooth muscles. Thus, they maintain a relatively constant diameter (Mbah, 2010). Katiyar , and Mbah (1996) studied the effect of time-dependent stenosis on the pulsatile flow through an elastic tube. It was discussed that the change in the height of the stenosis presents different velocities both radially and axially at a particular point of the stenosis. In general, the change in the height of the stenosis affects the velocities while at the same time exposes the cells of the walls to more serious pressure damage due to fluctuation.

Blood is non-Newtonian fluid and to model such fluid is very complicated. In this problem, blood is assumed to be a Newtonian fluid. Even though this will make the problem much simpler, it still is valid since blood in large vessel act almost like a Newtonian fluid. In order to model this problem, Navier-Stokes equations will be used to derive the governing equations that represent this problem (Labadin and Ahmadi, 2006).

This study will add value to research by not considering these specialized properties (like tapering, stenosis, smaller arterioles) but will describe blood flow in a typical artery in addition to showing variations in blood pressure through the systemic circulation.

2. METHODOLOGY

2.1. Model Construction

We shall consider the flow of blood through the large arteries. The artery is taken to be a cylindrical tube with elastic wall. Due to pumping of the heart (systole and diastole), the flow of blood through this vessel is pulsatile. The pulsatile motion of the blood leads to the deformation (movement of the wall in terms of dilation or contraction) of the material of the arterial wall. The motion of blood is in two forms namely the radial and axial flow which results from the influence of pressure. Here, we consider the flow to be steady and Newtonian. Thus, we shall construct a model describing the flow of blood through the aorta in which the fluid is Newtonian and incompressible. Since the wall of the artery is elastic, the motion of the blood is along the axis and radial direction (the angular direction is negligible), i.e. $U_z \neq 0$, $U_r \neq 0$ and =0.

The expression that will be used to describe this type of flow is the Navier-Stokes equation of motion of fluid flow. The solution to the equations describing this flow is in the form of Bessel's differential equations. Figure 1 shows a typical representation of an artery.

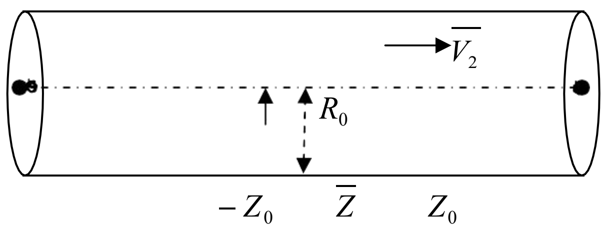

Fig 1: A diagrammatic representation of an artery

2.2. Assumptions of the Model

In order to obtain an appropriate model for the motion of blood flow, the following assumptions will be made:

1. Blood is considered as an incompressible Newtonian fluid.
2. The motion of the blood flow is assumed to be laminar.
3. The flow of blood in arteries is similar to the flow of blood in pipes with either rigid or elastic walls, where those with elastic walls can be considered more relevant here.
4. The angular velocity is negligible which was used to simplify the Navier-Stokes equation.
5. The Reynolds number (Re) is very small due to dominance of viscosity over shear stress in order that blood flow will not become turbulent.

2.3. Model of Blood Flow in the large arteries

The equations describing the velocity of blood flow both axially and radially together with the continuity equation are given as:

$$\frac{\partial \overline{U}_r}{\partial \bar{t}} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial \bar{r}} + \upsilon\left[\frac{\partial^2 \overline{U}_r}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \overline{U}_r}{\partial \bar{r}} - \frac{\overline{U}_r}{\bar{r}^2} + \frac{\partial^2 \overline{U}_r}{\partial \bar{z}^2}\right] \quad (1)$$

$$\frac{\partial \overline{U}_z}{\partial \bar{t}} = -\frac{1}{\rho}\frac{\partial P}{\partial \bar{z}} + \upsilon\left[\frac{\partial^2 U_z}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \overline{U}_z}{\partial \bar{r}} + \frac{\partial^2 \overline{U}_z}{\partial \bar{z}^2}\right] \quad (2)$$

$$\frac{\partial \overline{U}_z}{\partial \bar{z}} + \frac{\partial \overline{U}_r}{\partial \bar{r}} + \frac{\overline{U}_r}{\bar{r}} = 0 \quad (3)$$

where is the blood density, P is the blood pressure and is the kinematic viscosity of blood.

To avoid dimensional problem that might arise in subsequent calculation, we introduce the following non-dimensional variables:

$$U_r = \frac{\bar{U}_r}{U_0}, U_z = \frac{\bar{U}_z}{U_0}, P = \frac{\bar{P}}{\rho V_0}, t = \frac{[\bar{t}V_0]^2}{R_0}, r = \frac{\bar{r}}{R_0}, z = \frac{\bar{z}}{z_0}, \delta = \frac{\bar{\delta}(\bar{t})}{R_0}$$

Due to the pulsatile motion of the blood, the velocities and the pressure are all functions of r, z and t such that:

$$U_r = U_r(r,z,t) = U_1(r)e^{in\omega t - iy_n z} \quad (4)$$

$$U_z = U_z(r,z,t) = U_2(r)e^{in\omega t - iy_n z} \quad (5)$$

$$P = P(r,z,t) = P(r)e^{in\omega t - iy_n z}$$

$$where \quad y_n = \frac{2\pi}{\lambda_n} + i\delta(t) \quad (6)$$

Substituting equation (4), (5) and (6) into equations (1), (2) and (3), we have:

$$\frac{\partial^2 U_1}{\partial r^2} + \frac{1}{r}\frac{\partial U_1}{\partial r} - \left[y_n^2 + (inw - b)R_e + \frac{1}{r^2}\right]U_1 = R_e\left[\beta(2\, inw\, b) + \frac{\partial P(r)}{\partial r}\right] \quad (7)$$

$$\frac{\partial^2 U_2}{\partial r^2} + \frac{1}{r}\frac{\partial U_2}{\partial r} - \left[y_n^2 + (inw - b)R_e\right]U_2 = -iy_n R_e P(r) \quad (8)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rU) = iy_n^2 U_2 \quad (9)$$

$$where\, b = \frac{\delta_m z_n}{\tau}, but\, \delta_m = o$$ (maximum height attained by stenosis)

$$\delta(t) = \delta_m(1 - e^{\frac{t}{\tau}}) = 0$$

Let us for simplicity define $K_e^2 = y_n^2 + (inw-b)R_e$. Then we have:

$$\frac{\partial^2 U_1}{\partial r^2}+\frac{1}{r}\frac{\partial U_1}{\partial r}-\left[K_n^2+\frac{1}{r^2}\right]U_1 = K_e^2(inw-b)+\beta\frac{\partial P(r)}{\partial r} \qquad (10)$$

$$\frac{\partial^2 U_2}{\partial r^2}+\frac{1}{r}\frac{\partial U_2}{\partial r}-K_n^2 U_2 = iy_n K_e P(r)+\sigma(inw-b)K_e \qquad (11)$$

$$where\ \beta=\frac{\partial U_1(r)}{\partial t},\ \sigma=\frac{\partial U_2(r)}{\partial t}$$

Equation (10) and (11) are Bessel's type of differential equations. Hence, an assumption of solutions to this set of equations in the form of Bessel's functions will be rightly justified. Thus, we have the general solutions to these equations as:

$$Y=\beta_1^1 J_0(iy_n r)+\beta_2^1 J_0(ik_n r) \qquad (12)$$

$$X=\alpha_1^1 J_1(iy_n r)+\alpha_2^1 J_1(ik_n r) \qquad (13)$$

But since the axial and radial velocities are finite, we require that the expressions for X and Y be finite. Hence, we assume that $\alpha_2 = \beta_2 = 0$ for these expressions to be finite and hence substituting these new forms of these expressions we obtain:

$$X=\alpha_1^1 J_1(iy_n r)$$

$$Y=\beta_1^1 J_0(iy_n r)$$

Substituting these new expressions for X and Y in equation (10) and (11), expanding the Bessel functions, with Reynolds number (Re=1) we get that:

$$U_R=\sum_{n=0}^{\infty} R\alpha_1^1\left[\frac{y_n}{2}+\frac{y_n^3}{16}+\frac{y_n^5}{384}+\ldots\right]e^{in\omega t-iy_n z}$$

$$U_Z=\sum_{n=0}^{\infty} Q\beta_1^1\left[1+\frac{y_n^2}{4}+\frac{y_n^4}{64}+\ldots\right]e^{in\omega t-iy_n z}$$

$$P=\beta r(in\omega)e^{in\omega t-iy_n z}-\sum_{n=0}^{\infty}\frac{\alpha_1^1}{y_n}\left[(in\omega)\left(1+\frac{y_n^2}{4}+\frac{y_n^4}{64}+\ldots\right)\right]e^{in\omega t-iy_n z}$$

where

$$R = 2y_n^2 + in\omega$$
$$Q = y_n(in\omega)$$

3.1. ANALYSIS AND DISCUSSION OF RESULTS

The results were analyzed using the Matlab software. Deductions were made to enable the researcher give the necessary recommendations and infer conclusions to the study. This is illustrated below.

We shall consider a given point z=-0.01, λ =10, $r = 0.06$, $\omega = 2\pi f$, $f = 0.8$ (Katiyar et al, 1996). We shall consider the values of t from 0 to 0.9 with an increment of 0.01. Hence we have for n=1, the following:

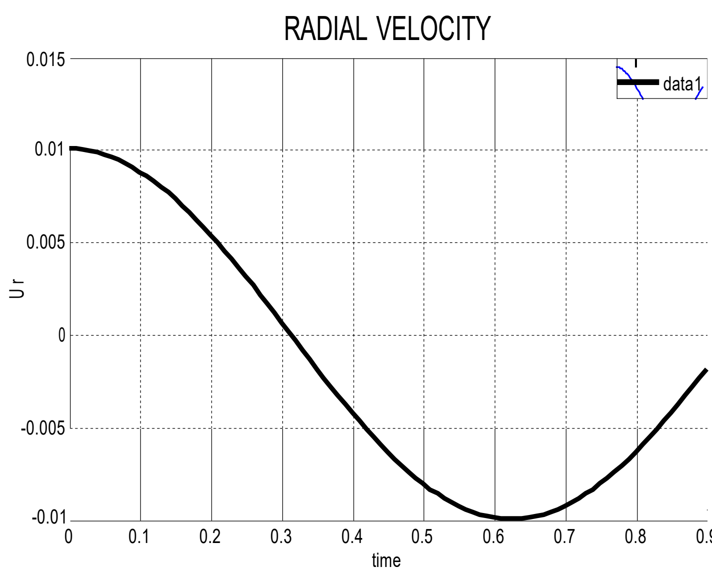

Fig. 2: Motion of blood flow in the radial direction

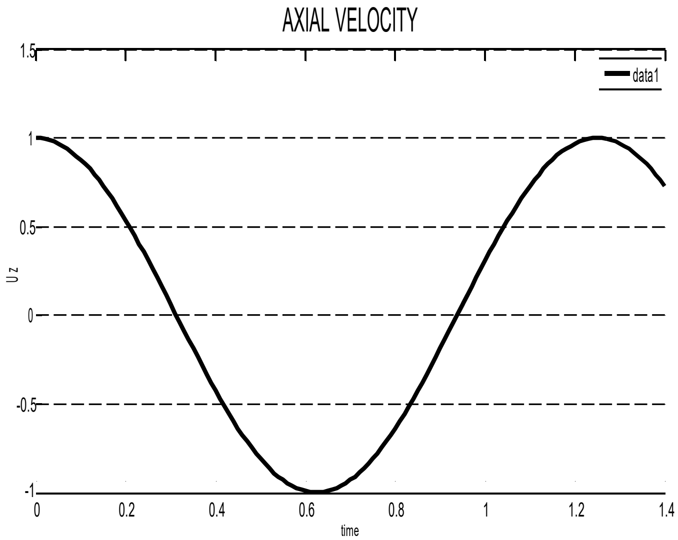

Fig. 3: Motion of blood flow in the axial direction

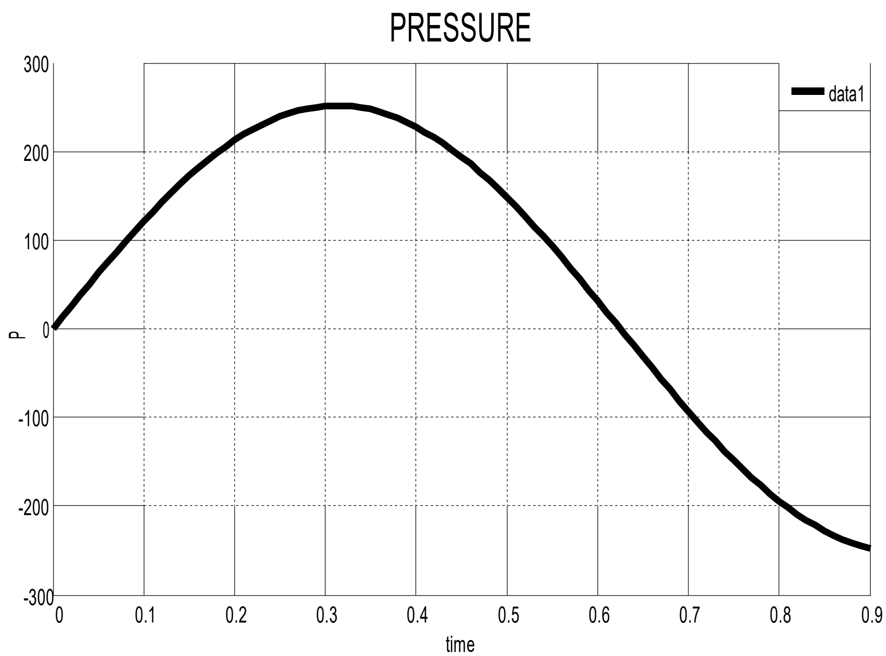

Fig. 4. Pressure gradient in blood flow

From the above values, we considered n=1 which describes uniform wave which is obtainable in the human system since n>1 represents number of waves which may not be obtainable in the human system. Here, we can see that blood flow in the radial and axial direction is both sinusoidal in motion. This is due to the motion of the heart (systole and diastole). The large arteries are very close to the heart so they respond the motion of the heart system. We can also see that radial and axial velocity stabilizes at a particular time (t=0.6) and begins to vary again.

We can also see that the blood pressure rises with time up to a certain point where it becomes steady and starts dropping. In agreement with (Guyton, 2006), the heart pumps blood continually into the aorta with a very high mean pressure of about 100 mm Hg. Moving further to where the large arteries divide to smaller arterioles and capillaries, the blood pressure falls progressively. As the blood flows through the *systemic circulation,* its mean pressure falls progressively to about 0 mm Hg by the time it reaches the termination of the venae cavae where they empty into the right atrium of the heart.

3.2. CONCLUSION

From the proposed study, we have seen the motion of blood flow in the radial and axial direction. We can therefore conclude that the pulsatile nature of blood flow leads to the wave-like motion of the axial and radial velocities. This is because the large arteries in consideration are close to the heart so they respond to the movement of the heart. This is also possible due to the elastic nature of the artery. For smaller arterioles farther away from the heart, it may not be obtainable.

We have also discovered the pressure differences at various points in circulation. In the circulatory system, pressure varies through the systemic and pulmonary circulation.

This research work provides a basic tool for health officers to work towards the optimal health of individuals. This helps to improve the standard of living of individuals and minimize cost of living; thus, it is of great advantage to the economy since the health and economic sector are not mutually exclusive. Rising health care spending can be viewed as both a weight on broader economic growth and as a driver of sectoral and local prosperity. Rapidly rising health care spending is considered to lower the rate of growth in GDP and overall employment, while raising inflation.

REFERENCES

Thibodeau, G. A., Patton, K. T., *Anatomy and Physiology*. 4th ed., Mosby, St Louis, 1999.

Martini, F. H., *Fundamentals of Anatomy and Physiology*. Prentice Hall, Englewood Cliffs, N. J., 1995.

J. Labadin, A. Ahmadi (2006), Mathematical Modeling of the Arterial Blood Flow, Regional conference on Mathematics, Statistics and Applications, Universiti Sains Malaysis, Penang.

Mbah, G.C.E (2010), *Introduction to Mathematical modeling*. Undergraduate LectureNote, University of Nigeria, Nsukka.

Guyton, A. C., and Hall J. E., *Textbook of Medical Physiology*, 11th ed., W. B. Saunders, Philadelphia, 2006.

Katiyar V, Mbah,G, (1996), *Effect of time-dependent stenosis on the pulsatile flow through an elastic tube:* Journal of Mathematical sciences, Department of Mathematics, University of Roorkee, India.